Semiring schemes

I Semirings

semiring = set R with elements $0, 1 \in R$ and binary operations $+$ and $\cdot$ such that

- $(R, +, 0)$ is a commutative monoid
- $(R, \cdot, 1)$ is a commutative monoid
- $0 \cdot a = 0$
- $a(b+c) = ab + ac$

for all $a, b, c \in R$.

semiring morphism = map $f: R \to R$ s.t. for all $a, b \in R$

- $f(0) = 0$, $f(1) = 1$
- $f(a+b) = f(a) + f(b)$
- $f(ab) = f(a) \cdot f(b)$

$\rightsquigarrow$ category **SRings**

Fact: SRings is complete, cocomplete and inner.

$\left(\rightsquigarrow R_1 \otimes_{R_0} R_2 = \operatorname{colim} \left\{ R_0 \begin{smallmatrix} \nearrow R_1 \\ \searrow R_2 \end{smallmatrix} \right\} \right)$

unit group $R^\times = \{ a \in R \mid ab = 1 \text{ for some } b \in R \}$

semifield = semiring k s.t. $k^\times = k - \{0\}$

Fact: If $0 = 1$, then $R = \{0\}$.

Ex: (0) Rings

(1) $\mathbb{N}$ with $\mathbb{N}^\times = \{1\}$ (initial object in SRings)

(2) $\mathbb{R}_{\geq 0}$ semifield

(3) $\mathbb{B} = (\{0,1\}, \max, \cdot)$ (Boolean semifield)

(4) $\mathbb{T} = (\mathbb{R}_{\geq 0}, \max, \cdot)$ (tropical semifield)

(5) $\mathcal{O}_{\mathbb{T}} = ([0,1], \max, \cdot)$ (tropical integers)

II Ideals

ideal = subset $I \subset R$ s.t. for all $a, b \in I$, $c \in R$

- $0 \in I$
- $a + b \in I$
- $ca \in I$

prime ideal = ideal $p \subset R$ s.t. $S = R - p$ is a multiplicative set ($1 \in S$ and $a, b \in S \Rightarrow ab \in S$).

proper ideal = ideal $I \neq R$

maximal ideal = proper ideal that is maximal w.r.t. inclusion

Facts: • If R is nontrivial, it has a maximal ideal.

• Every maximal ideal of R is prime.

Ex: (1) k semifield $\rightsquigarrow$ ideals $\{0\}$ & k

$\{0\}$ prime & max.

(2) $\mathbb{N} \rightsquigarrow$ ideal = additive submonoid

$$= \langle a_i \rangle_{i\in I} = \Big\{ \textstyle\sum u_i a_i \;\Big|\; (u_i) \in \bigoplus_{i\in I} \mathbb{N} \Big\}$$

$\uparrow$ (always finitely generated)

prime ideals

$$\begin{tikzcd}[column sep=small]
 & & \mathfrak{m} = \langle 2,3\rangle = \mathbb{N} - \mathbb{N}^\times \arrow[dll, no head] \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & & \\
\langle 2\rangle \arrow[drr, no head] & \langle 3\rangle \arrow[dr, no head] & \langle 5\rangle \arrow[d, no head] & \langle 7\rangle \arrow[dl, no head] & \cdots \\
 & & \langle 0\rangle & &
\end{tikzcd}$$

Local semiring = semiring with a unique maximal ideal

Fact: R local $\Rightarrow$ $\mathfrak{m} = R - R^\times$ maximal ideal

III Monoid algebras

A monoid (commutative, multiplicatively written)

R semiring

$$R[A] = \Big\{ \sum_{a\in A} r_a \cdot a \;\Big|\; (r_a) \in \bigoplus_{a\in A} R \Big\}$$ (monoid algebra of A over R)

with
$$\left(\sum r_a \cdot a\right) + \left(\sum s_a \cdot a\right) = \sum (r_a + s_a)\cdot a$$
$$\left(\sum r_a \cdot a\right) \cdot \left(\sum s_a \cdot a\right) = \sum \Big(\sum_{bc=a} r_b \cdot s_c\Big)\cdot a$$

universal property

$\forall\, \varphi_R : R \to R'$, $\varphi_A : A \to (R', \cdot)$

$\exists!\, \hat\varphi : R[A] \to R'$ s.t.

$$\begin{tikzcd}
R \arrow[dr, "r \mapsto r\cdot 1"'] \arrow[drr, bend left, "\varphi_R"] & & \\
 & R[A] \arrow[r, "\hat\varphi"] & R' \\
A \arrow[ur, "a \mapsto 1\cdot a"] \arrow[urr, bend right, "\varphi_A"'] & &
\end{tikzcd}$$

Ex: Polynomial algebra

$$R[T_1, \dots, T_n] = \left\{ \sum r_{e_1 \dots e_n} T_1^{e_1} \cdots T_n^{e_n} \right\} = R[A]$$

for $A = \{ T_1^{e_1} \dots T_n^{e_n} \mid e_1, \dots, e_n \geq 0 \}$.

IV Quotients

In general, quotients of semirings do not correspond to ideals:

- $\ker \left(\mathbb{N} \to \mathbb{B} \right) = \{0\}$
  $a \mapsto \begin{cases} 0, & a=0 \\ 1, & a>0 \end{cases}$

- $\mathbb{N}/\langle 2,3 \rangle = \{0\}$ since $1 + \langle 2,3 \rangle \equiv 3 + \langle 2,3 \rangle \equiv 0 + \langle 2,3 \rangle$.

congruence = equivalence relation $\sim$ on R s.t. $\forall a,b,c \in R$
$a \sim b \Rightarrow a+c \sim b+c$ and $ac \sim bc$.
(= subsemiring of $R \times R$ that is an equiv. rel.)

$\rightsquigarrow$ $R/\sim$ semiring with $\cdot\ [a]+[b]=[a+b]$
$\cdot\ [a]\cdot[b]=[ab]$

Fact: $\left\{ \begin{matrix} \text{congruence} \\ \text{on } R \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{quotients} \\ R \twoheadrightarrow R' \text{ (mod iso)} \end{matrix} \right\}$

$\sim \quad \longmapsto \quad R \twoheadrightarrow R/\sim$

$\operatorname{congker}(\pi) = \{(a,b) \mid \pi(a) = \pi(b)\} \quad \longleftarrow \quad R \xrightarrow{\pi} R'$

$\left(= \operatorname{eq}\left(R \times R \underset{\pi}{\overset{\pi}{\rightrightarrows}} R' \right) \right)$

$S \subset R \times R \rightsquigarrow \langle S \rangle = \left(\begin{array}{l}\text{smallest congruence} \\ \text{containing } S\end{array}\right)$

<u>universal property</u>

$S \subset R \times R$, $\forall \varphi : R \to R'$ s.t. $\forall (a,b) \in S$, $\varphi(a) = \varphi(b) \in R'$

$\exists!\ \bar{\varphi} : R/_{\langle S \rangle} \longrightarrow R'$ s.t.

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\pi"'] & R' \\ R/_{\langle S \rangle} \arrow[ur, "\bar{\varphi}"'] & \end{tikzcd}$$

<u>Fact:</u> $\{\text{ideals on } R\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{\text{congruences on } R\}$

$I \longmapsto \sqsubset_I = \langle (a,0) \mid a \in I \rangle$

$I_\sqsubset = \ker(R \to R/_\sqsubset) \longleftarrow\!\shortmid\ \sqsubset$

satisfy: $\cdot\ \Phi \circ \Psi(\sqsubset_I) = \sqsubset_I$

$\cdot\ \Psi \circ \Phi(I_\sqsubset) = I_\sqsubset$ (k-ideal)

(I k-ideal iff. $a, a+b \in I \Rightarrow b \in I$)

V Localizations

$S \subset R$ multiplicative set

$S^{-1}R = S \times R /\sim$ for $(s,a) \sim (s',a')$ iff. $\exists t \in S$ s.t. $tsa' = ts'a$

with $\frac{a}{s} + \frac{b}{t} = \frac{ta+sb}{st}$ and $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$

where $\frac{a}{s} = [(s,a)]$.

universal property:

$S \subset R$ mult. set

$\forall f: R \to R'$ s.t. $f(S) \subset (R')^\times$

$\exists! \hat{f}: S^{-1}R \to R'$ s.t.

$$\begin{array}{c} a \\ \updownarrow \\ \frac{a}{1} \end{array} \qquad \begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\iota_S"'] & R' \\ S^{-1}R \arrow[ur, "\hat{f}"'] \arrow[u, phantom, "\text{''}\circlearrowleft\text{''}"] & \end{tikzcd}$$

Fact: • $\iota_S: A \hookrightarrow S^{-1}A$ is injective if A is integral

($ab = ac \Rightarrow a = 0$ or $b = c$) (also: (mult.) cancellative $\Leftrightarrow \exists A \hookrightarrow$ semifield)

Rem: $\mathbb{B}[T]$ is without zero divisors ($a.b = 0 \Rightarrow a = 0$ or $b = 0$)

but not integral. $(T+1)(T^2+1) = T^3+T^2+T+1$
$= (T+1)(T^2+T+1)$

• $p \subset R$ prime ideal $\rightsquigarrow R_p = S^{-1}R$ for $S = R - p$

• $h \in R \rightsquigarrow R[h^{-1}] = S^{-1}R$ for $S = \{h^i\}_{i \in \mathbb{N}}$

VI The spectrum

semiringed space = topological space X
+ sheaf $\mathcal{O}_X$ in SRings

Local morphism = continuous map $\varphi: X \to Y$
+ sheaf morphism $\varphi^\#: \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$
s.t. $\forall x \in X$, $y = \varphi(x)$, $\varphi_x: \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$
sends non-units to non-units.

Spec R = semiringed space $X = \{p \subset R \text{ prime ideal}\}$

+ Zariski topology generated by

principal opens

$$U_h = \{ p \mid h \notin p \} \qquad (h \in R)$$

+ structure sheaf $\mathcal{O}_X$ with

- $\mathcal{O}_X(U_h) = R[h^{-1}]$
- $\mathcal{O}_{X,p} = R_p$

Thm (Toën-Vaquié, Marty, +ε)

There is a (unique) sheaf $\mathcal{O}_X$ with $\mathcal{O}_X(U_h) = R[h^{-1}]$ (and $\mathcal{O}_{X,p} = R_p$).

(explanations later)

- $p \in X = \operatorname{Spec} R$, $m_p = R_p - R_p^\times$

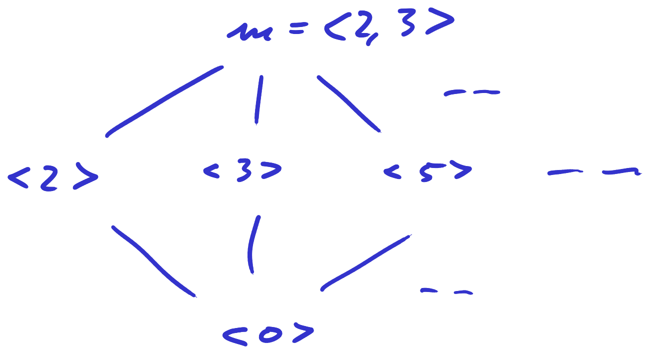
residue field of p: $k(p) = R_p / m_p$

Fact: $k(p) \neq \{0\}$ $\iff$ p is a k-ideal

$\iff$ $k(p)$ semifield

Ex: (1) k semifield $\rightsquigarrow X = \operatorname{Spec} k$ $\qquad \langle 0 \rangle \qquad k(0) = k$

(2) $\operatorname{Spec} \mathbb{N}$



$k(\mathfrak{m}) = \{0\}$

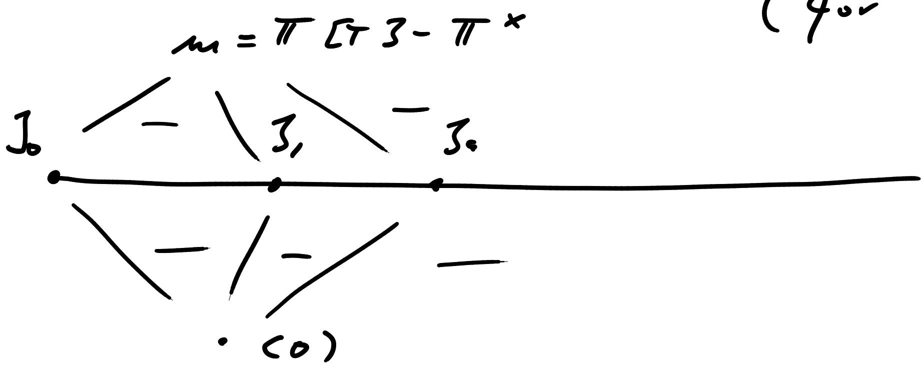
$k(\langle p \rangle) = \mathbb{F}_p$

$k(\langle 0 \rangle) = \mathbb{Q}_{\geq 0}$

(3) $\operatorname{Spec} \mathbb{T}[T]$ $\qquad J_a = \left\langle \sum c_i T^i \,\middle|\, \begin{matrix} \text{max occur} \\ \text{in } \{c_i a^i\} \end{matrix} \text{ twice} \right\rangle$ (for $a \in \mathbb{T}$)

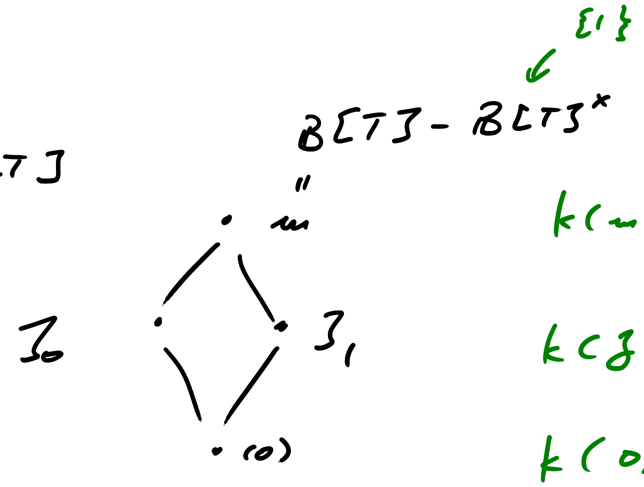
$k(\mathfrak{m}) = \mathbb{T}$

$k(J_a) = \mathbb{T}$



$k(0)$
$\downarrow$
$\operatorname{Fun}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$

(4) $\operatorname{Spec} \mathbb{B}[T]$



$k(\mathfrak{m}) = \mathbb{B}$

$k(J_a) = \mathbb{B}$

$k(0) = \mathbb{B}$

VII Semiring schemes

- **affine semiring scheme** = semiringed space isomorphic to Spec R for some R

- **semiring scheme** = semiringed space X s.t. $\exists$ open covering $X = \bigcup_{i\in I} U_i$ by affine semiring schemes U_i (for $\mathcal{O}_{U_i} = \mathcal{O}_X|_{U_i}$)

$\rightsquigarrow$ category $Sch_{\mathbb{N}}$ (final object Spec $\mathbb{N}$)

- $Sch_R = \left\{ X \to \operatorname{Spec} R \text{ + } \begin{array}{c} X \to Y \\ \searrow \ \circlearrowleft \ \swarrow \\ \operatorname{Spec} R \end{array} \right\}$

<u>Fact</u>: (category of schemes) $\simeq Sch_{\mathbb{Z}} \xrightarrow[\text{faithful}]{\text{fully}} Sch_{\mathbb{N}}$

(① a semiring R is a ring $\Longleftrightarrow \exists\, \mathbb{Z} \to R$
② Spec R is the usual Spec for rings)

<u>Prop</u>: $\text{SRings} \xrightarrow{\text{Spec}} Sch_{\mathbb{N}}$ is a reflection, with left adjoint Γ $\quad (\Gamma X = \Gamma(X, \mathcal{O}_X))$

i.e. • $\Gamma \operatorname{Spec} R = R$

• $\operatorname{Hom}(X, \operatorname{Spec} R) = \operatorname{Hom}(R, \Gamma X)$

for all $X \in Sch_{\mathbb{N}}$, $R \in$ SRings

VIII Semiring schemes from monoid schemes

- k semiring

  A pointed monoid, $A_k^+ = k[A] / \langle 0_A \sim 0_{k[A]} \rangle$

  $X = \operatorname{Spec} A$ ("$= A \otimes_{\mathbb{F}_1} k$")

  $X_k^+ = \operatorname{Spec} A_k^+$ ("base extension to semiring schemes$/k$")

- X monoid scheme

  $X = \operatorname{colim} \left\{ \begin{array}{l} U \subset X \text{ affine} \\ + \text{ principal opens } V \hookrightarrow U \end{array} \right\}$

  $X_k^+ = \operatorname{colim} \{ U_k^+ + V_k^+ \hookrightarrow U_k^+ \}$

- $X^+ = X_{\mathbb{N}}^+$ (Fact: $A[h^{-1}]_k^+ = A_k^+[h^{-1}]$)

Ex: (1) $\mathbb{A}_{\mathbb{N}}^n = \operatorname{Spec} \mathbb{N}[T_1 - T_n] = \mathbb{A}_{\mathbb{F}_1}^{n,+}$ $(= \operatorname{Spec} \mathbb{F}_1[T_1 - T_n])$

(2) Δ fan in $\mathbb{R}^n$, k semiring

$\sigma \in \Delta \rightsquigarrow A_\sigma = \sigma^\vee \cap \mathbb{Z}^n$

$U_\sigma = \operatorname{Spec} k[A_\sigma]$

$X_k(\Delta) = \operatorname{colim} \{ U_\sigma + \text{inclusions} \} = Y_k^+$

(toric k-variety) for $Y = \operatorname{colim} \{ \operatorname{Spec}(A_\sigma \cup \{0\}) \}$

IX Functor of points

- Typically, the underlying topological space of a semiring scheme is not very interesting, but rational point sets are.

- k semiring

  $X \in Sch_k$

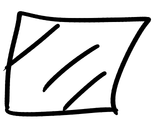
  $k \to R$

  $$X(R) = Hom_k(Spec\, R, X) \quad \text{(R-rational points of X)}$$

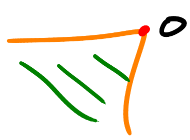
Ex: $R = k = \mathbb{T}$

(1) $$X = \mathbb{G}^n_{m,\mathbb{T}} = Spec\, \underbrace{\mathbb{T}[T_1^{\pm 1} \_ T_n^{\pm 1}]}_{= \mathbb{T}[T_1, - T_n][T_1^{-1}, -, T_n^{-1}]}$$

$$\rightsquigarrow \mathbb{G}^n_{m,\mathbb{T}}(\mathbb{T}) = Hom_{\mathbb{T}}(\mathbb{T}[T_1^{\pm 1} \_ T_n^{\pm 1}], \mathbb{T}) = (\mathbb{T}^\times)^n$$



$$\mathbb{R}^n_{>0} \underset{\text{(homeo)}}{\simeq} (0,1)^n \underset{\text{(homeo)}}{\simeq} \text{(open unit disc of dim. n)}$$

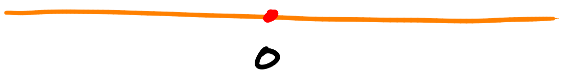
(2) $\Delta =$ 

$$\rightsquigarrow X_{\mathbb{T}}(\Delta) = Spec\, \mathbb{T}[T_1, T_2] = \mathbb{A}^2_{\mathbb{T}}$$

$$X_{\mathbb{T}}(\Delta)(\mathbb{T}) = Hom_{\mathbb{T}}(\mathbb{T}[T_1, T_2], \mathbb{T}) = \mathbb{T}^2$$

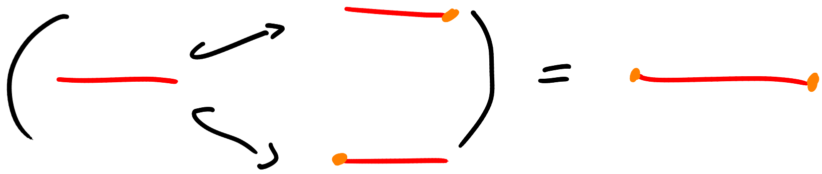
$\mathbb{T}^2 = \mathbb{R}^2_{\geq 0}$

(3) Δ

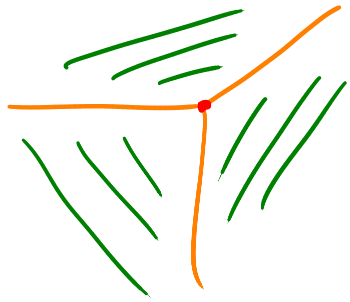


$X_\pi(\Delta) = \mathbb{P}^1_\pi = \mathbb{A}^1_\pi \amalg_{\mathbb{G}_{m,\pi}} \mathbb{A}^1_\pi$

$\mathbb{P}^1_\pi(\pi) = \operatorname{colim}$

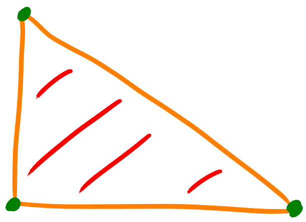


(4) $\Delta =$



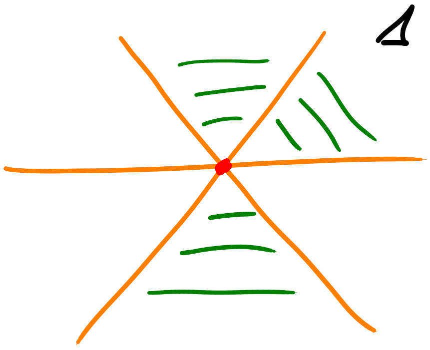
$\rightsquigarrow X_\pi(\Delta) = \mathbb{P}^2_\pi$

$\mathbb{P}^2_\pi(\pi) =$

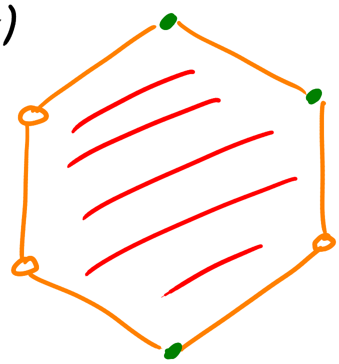


<u>Rem:</u> $X_\pi(\Delta)(\pi)$ is a partial topological compactification of $\mathbb{R}^n$; e.g.

Δ $\qquad$ $X_\pi(\Delta)(\pi)$



$\rightsquigarrow$



X Relation to semiring schemes à la Toën-Vaquié

→ $\mathcal{A}$ category (of "algebraic objects")

$$\mathcal{A} \underset{\Gamma}{\overset{\mathrm{Spec}}{\underset{\sim}{\rightleftarrows}}} \mathcal{A}^{op} =: \mathrm{Aff}_{\mathcal{A}} \quad (\text{affine } \mathcal{A}\text{-schemes})$$

(anti-equivalence)

→ A Grothendieck pretopology $\mathcal{T}$ on $\mathrm{Aff}_{\mathcal{A}}$

is a collection of covering families $\{U_i \to X\}_{i\in I}$

s.t. • $\{id : X \to X\}$ is a covering family

• stable under pullbacks & refinements

→ $\mathcal{T}$ is subcanonical if for every $X \in \mathrm{Aff}_{\mathcal{A}}$

$h_X = \mathrm{Hom}(-, X)$ is a sheaf on the site $(\mathrm{Aff}_{\mathcal{A}}, \mathcal{T})$,

i.e. $\forall$ covering family $\{U_i \to X\}$,

$\forall\, Y \in \mathrm{Aff}_{\mathcal{A}}$,

$$U_{ij} = U_i \times_X U_j \begin{array}{l} \xrightarrow{\pi_i} U_i \\ \xrightarrow{\pi_j} U_j \end{array}$$

$$\mathrm{colim}\left\{ \mathrm{Hom}(Y, U_{ij}) \begin{array}{l} \xrightarrow{\pi_{i,*}} \\ \xrightarrow{\pi_{j,*}} \end{array} \begin{array}{l} \mathrm{Hom}(Y, U_i) \\ \mathrm{Hom}(Y, U_j) \end{array} \right\} \longrightarrow \mathrm{Hom}(Y, X)$$

is a bijection.

Rem: There is a "finest" subcanonical Grothendieck pretopology:

the canonical topology $\mathcal{T}_{can}$.

__Yoneda Lemma:__ If J is subcanonical, then

$$\mathrm{Aff}_A \hookrightarrow \mathrm{Sh}(\mathrm{Aff}_A, J)$$
$$X \longmapsto h_X = \mathrm{Hom}(-, X)$$

is fully faithful.

$\to$ __principal open__ = $U \to X$ that appears in some covering family $\{U_i \to X\}$

__A-scheme__ = colimit of a "monodromy-free" diagram of principal opens in $\mathrm{Sh}(\mathrm{Aff}_A, J)$

(no self-identifications

__eg.__
$$\begin{tikzcd} & V \arrow[dl, "id"'] \arrow[dr, "id"] & \\ U & & U \\ & V \arrow[ul, "id"] \arrow[ur, "flip"'] & \end{tikzcd}$$
)

$\to$ __Toën - Vaquié:__ (simplified account)

$$\mathrm{Aff}_{\mathbb{N}} = \mathrm{SRings}^{op}$$

$$J_0 = \left\{ \{\mathrm{Spec}\, R[l_i^{-1}] \longrightarrow \mathrm{Spec}\, R\}_{i\in I} \;\middle|\; \langle l_i \rangle_{i\in I} = \langle 1 \rangle \right\}$$

__Rem:__ · $\{\mathrm{Spec}\, R[l_i^{-1}] \to \mathrm{Spec}\, R\}_{i\in I}$ is in J_0 iff it is in J_{can}.

$\Rightarrow$ J_0 is the finest subcanonical topology generated by principal opens.

$\mathbb{N}$-scheme = SRings-scheme (w.r.t. $\mathcal{J}_0$)

open immersions = $\varphi: U \to X$ in Sch_{SRings} s.t. $\forall \alpha: \operatorname{Spec} R \to X$ the base change

$$\pi_2: U \times_X \operatorname{Spec} R \longrightarrow \operatorname{Spec} R$$

is a monomorphism and

$$\operatorname{im}(\pi_2) = \operatorname{im}\Big(\coprod_{i \in I} \operatorname{Spec} R[h_i^{-1}] \to \operatorname{Spec} R\Big)$$

for some $\{h_i\}_{i \in I} \subset R$.

This defines the Zariski topology $\mathcal{J}_{Zar}$ on Sch_{SRings}.

Question: Is $\{U \to X \text{ open}\}/\sim$ the locale of opens of a topological space?

Thm (Marty)

If $X = \operatorname{Spec} R$, then $\{U \to X \text{ open}\}/\sim$ is the locale associated with

$\{\mathfrak{p} \subset R \text{ prime ideal}\}$ + Zariski topology.

Consequence: $Sch_{SRings} \simeq Sch_{\mathbb{N}}$.